Mathematics of Imaging & AI

Sparse optimization for discretized PDE-constrained problems with total variation regularization

Giacomo Cristinelli

Joint work with:
José A. Iglesias (University of Twente)
Daniel Walter (HU Berlin)

Schiermonnikoog 2023

Motivation, challenges and goals

Let $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ be a bounded Lipschitz domain.

$$\min_{u \in BV(\Omega)} F(Ku) + \alpha TV(u, \Omega) \tag{1}$$

- **Applications**: imaging, optimal control, ...

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- **Applications**: imaging, optimal control, ...
- **Numerical issues**: K is usually costly to compute, TV is non-smooth.
- **Goal**: offer a grid method to approximate (1) using
  - $\rightarrow$ Generalized conditional gradient (construction from levelset/extremal points)
  - $\rightarrow$ Graph cuts (to find the levelset/extremal points)

Formulation of the optimal control problem

Let $\Omega \subset \mathbb{R}^d$ with $d = 2, 3$ be a connected bounded domain.

$$\min_{u \in V} \frac{1}{2} \|Ku - y_d\|^2_{L^2(\Omega)} + \alpha TV(u, \Omega) \tag{$\mathcal{P}$}$$

where

- $V \subset L^q(\Omega)$ is a weakly closed linear subspace, with $q = \frac{d}{d-1}$, e.g.
  - $\rightarrow$ $BV(\Omega) \hookrightarrow L^q(\Omega)$ continuously embeds in $L^q(\Omega)$;
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- $\alpha > 0$, and $y_d \in L^2(\Omega)$;
- $K : V \rightarrow H^1(\Omega)$ is the *control to state operator* associated to

$$\begin{cases} -\operatorname{div}(A\nabla y) + cy = u & \textit{in } \Omega \\ y = 0 & \textit{on } \partial\Omega. \end{cases}$$

Existence of minimisers

Minimization problem:

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Auxilliary problem: $\mathring{V} \cong V/\{TV = 0\}$

$$\min_{u \in \mathring{V}} \mathcal{F}(Ku) + \alpha TV(u, \Omega) \tag{$\mathcal{Q}$}$$

where

$$\mathcal{F}(y) = \min_{c \in \mathbb{R}} \frac{1}{2} \|y + cK\mathbb{1}_\Omega - y_d\|^2_{L^2(\Omega)}$$

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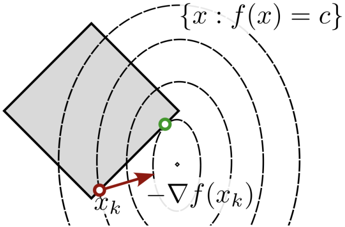
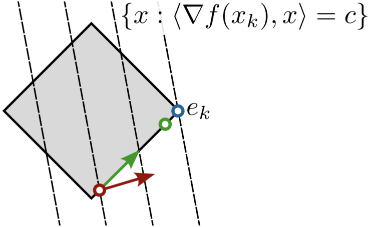
If $\mathbb{1}_\Omega \notin \ker(K)$, then $(\mathcal{P})$ admits a unique minimiser.

Conditional gradient methods

- Conditional gradient: (Frank, Wolfe '56) $\rightarrow$ $\min_{x \in C} f(x)$





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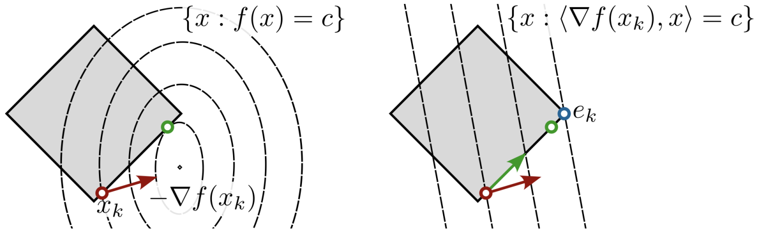

- Generalized conditional gradient: (Bredies, Lorenz, Maass 2009)

  $\rightarrow$ Given $u_k \in \mathring{V}$ and $p_k = K^* \nabla \mathcal{F} K u_k$, solve

$$v_{k+1} \in \operatorname*{arg\,min}_{v \in Ext(\alpha TV \leq 1)} \langle p_k, v \rangle$$

and update $u_{k+1} = u_k + t_{k+1}(v_{k+1} - u_k)$, for some $t_{k+1} \in \mathbb{R}$.

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- $\rightarrow$ it enforces properties of the regulariser (piecewise constant functions)
- $\rightarrow$ Clear description of the extremal points of TV-balls:

Lemma (Bredies, Carioni - C., Iglesias, Walter)

Let V be either $BV(\Omega)$ or $P_0(\Omega, \mathcal{T})$. For $\mathring{B} = \{u \in \mathring{V} : \alpha TV(u, \Omega) \leq 1\}$, then

$$Ext(\mathring{B}) = \left\{ \frac{\mathring{\mathbb{1}}_E}{\alpha P(E, \Omega)} : E \subset \Omega \text{ is simple} \right\}$$

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→ Connection to the prescribed mean curvature problems:

$$\min_{E \subset \Omega} P(E, \Omega) - \frac{1}{\alpha} \int_E p_k \, dx$$

Fully corrective generalized conditional gradient

Let $\mathcal{A} = \{v_i\}_{i=1}^N \subset Ext(\mathring{B})$ be the *active set* for u_k, i.e. $u_k = c\mathbb{1}_\Omega + \sum_{i=1}^N \gamma_i v_i \in V$.

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Update: Solve

$$(\overline{c}, \overline{\gamma}) \in \operatorname*{arg\,min}_{(c,\gamma) \in \mathbb{R} \times \mathbb{R}_+^{N+1}} \left[\frac{1}{2} \left\| \sum_{i=1}^{N+1} \gamma_i K v_i + cK\mathbb{1}_\Omega - y_d \right\|_{L^2(\Omega)}^2 + |\gamma|_1 \right]$$

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and update

$$u_{k+1} = \overline{c}\mathbb{1}_\Omega + \sum_{i=1}^{N+1} \overline{\gamma_i} v_i, \qquad \mathcal{A} = \{v_i : \overline{\gamma_i} \neq 0\}$$

Properties of the dual variable

Given $u_k \in V$ of the form $u_k = \mathring{u}_k + c_k \mathbb{1}_\Omega$ with $\mathring{u}_k \in \mathring{V}$. Then,

$$p_k := K^* \nabla \mathcal{F}(K\mathring{u}_k) = K^* \nabla F(Ku_k)$$

In particular,

$$c_k = \operatorname*{arg\,min}_{c \in \mathbb{R}} \|K\mathring{u}_k + cK\mathbb{1}_\Omega - y_d\|^2_{L^2(\Omega)} \qquad \Longleftrightarrow \qquad \int_\Omega p_k = 0$$

If $\bar{u} \in V$ is a minimiser of $(\mathcal{P})$, then $\bar{p} \in \partial TV(\bar{u}) \subset \partial TV(0)$, where

$$\partial TV(0) = \{\operatorname{div} T : T \in L^\infty(\Omega, \mathbb{R}^n), \|T\|_{L^\infty} \leq 1, T \cdot \nu_\Omega = 0 \text{ on } \partial\Omega\}.$$

Insertion step: generalized Cheeger sets

$$\eta = \min_{|E|>0,\ simple} \frac{1}{P(E,\Omega)} \int_E p_k \, dx \leq 0 \qquad (\mathcal{I})$$

Since p_k is not positive almost everywhere, there exists a minimiser and $\eta < 0$.

Difficulties: Existence of local minima.

Numerical approaches:

- Primal dual and threshold (Castro, Duval, Petit 2023)
- Dinkelbach: (C., Iglesias, Walter 2023+)

$$E_{n+1} \in \arg\min_{E \subset \Omega} \left[P(E,\Omega) - \lambda_n \int_E p_k \, dx \right], \qquad \lambda_n = \left(\frac{1}{P(E_n,\Omega)} \int_{E_n} p_k \, dx \right)^{-1}$$

Dinkelbach: prescribed mean curvature problem

Consider the map $\mathcal{G} : (-\infty, 0] \to \mathbb{R}$

$$\lambda \mapsto \min_{E \subset \Omega} \left[P(E, \Omega) - \lambda \int_E p_k \, dx \right]$$

Lemma

The map $\mathcal{G}$ satisfies:

- *For all $\lambda \leq 0$, we have that $\mathcal{G}(\lambda) \leq 0$;*
- *$\mathcal{G}(\lambda) = 0$ if and only if $\lambda \in \left[\frac{1}{\eta}, 0\right]$.*
- *$\mathcal{G}$ is strictly increasing for $\lambda \in \left(-\infty, \frac{1}{\eta}\right)$.*

Dinkelbach: sets of prescribed curvature

Given (E_n, λ_n) with $\lambda_n < \frac{1}{\eta}$ and E_n being a minimiser of $\mathcal{G}(\lambda_n)$. Solve

$$E_{n+1} \in \arg\min_{E \subset \Omega} \left[Per(E, \Omega) - \lambda_{n+1} \int_E p_k \, dx \right], \qquad \lambda_{n+1} = \left(\frac{1}{P(E_n, \Omega)} \int_{E_n} p_k \, dx \right)^{-1}$$

Lemma (convergence)

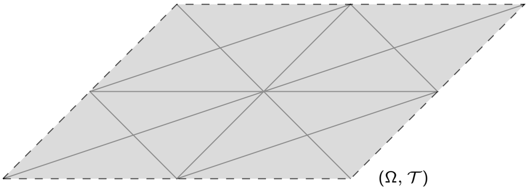
Let $\overline{E}$ be a minimiser of $(\mathcal{I})$. For the sequence $(E_n, \lambda_n)_n$ as before,

$$0 \leq \left| \eta - \frac{1}{\lambda_{n+1}} \right| \leq \left| \eta - \frac{1}{\lambda_1} \right| \left(1 - \frac{P(\overline{E}, \Omega)}{P(E_1, \Omega)} \right)^n$$

Initialization: $\lambda_1 = -\frac{1}{\alpha}$, and $E_1 \in \arg\min_{E \in S_{\mathcal{T}}(\Omega)} \left[P(E, \Omega) - \lambda_1 \int_E p_k \, dx \right]$

Graph cuts: general idea

Let $(\Omega, \mathcal{T})$ be a triangulated domain.



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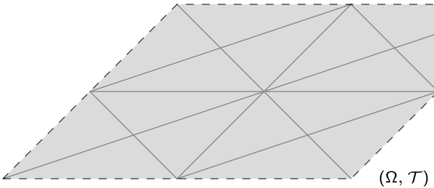

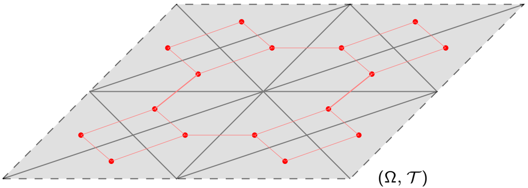

Each face is a node of the graph, and there is an edge only between nodes of adjacent triangles.

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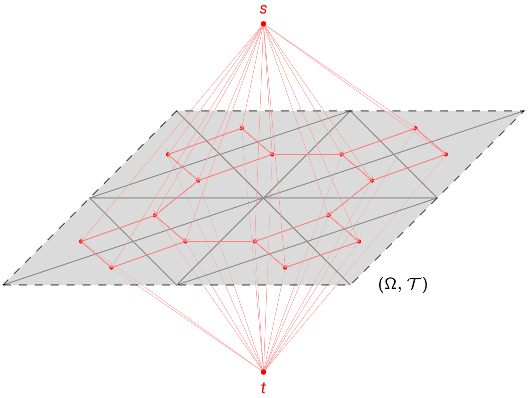

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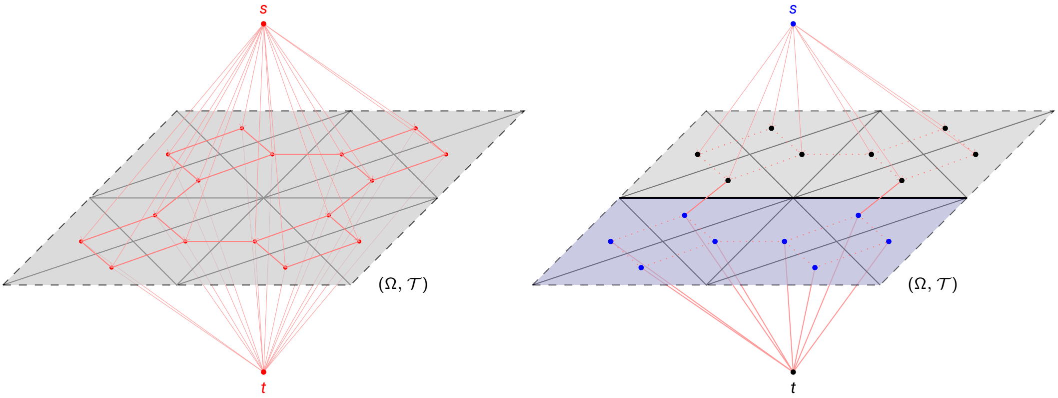


$$P_1 := \{N \in blue\} \cup \{s\} \text{ and } P_2 := \{N \in black\} \cup \{t\}$$
$$\theta_{cut} = \sum_{e \in red} w(e)$$

Graph cut for discrete mean curvature problems

Let $(\Omega, \mathcal{T})$ be a triangulated domain, and construct the (s,t)-graph as before. We define the weights as

$$w_{i,j} := \mathcal{H}^{d-1}(\partial T_i \cap \partial T_j)$$
$$w_{s,i} := \max\{0, \lambda p_k(T_i)\mathcal{L}^d(T_i)\}$$
$$w_{i,t} := \max\{0, -\lambda p_k(T_i)\mathcal{L}^d(T_i)\}.$$

Lemma

Given a (s,t)-cut given by $I \cup \{s\}$ and $I^c \cup \{t\}$, the cut value corresponds to

$$\theta_{cut} = P(E_I, \Omega) - \lambda \int_{E_I} p_k \, dx + \int_{\Omega} \max\{0, \lambda p_k\} \, dx$$

where $E_I \subset \Omega$ is the union of triangles corresponding to I.

Numerical example: the castle

Goal: Let $\Omega = [-1,1]^d$, and $y_0 = \mathbb{1}_{[-1/2,1/2]^d} \in L^2(\Omega)$. Minimize

$$\min_{u \in L^2(\Omega)} \frac{1}{2}\|y - y_0\|^2_{L^2(\Omega)} + \alpha TV(u,\Omega)$$

subject to

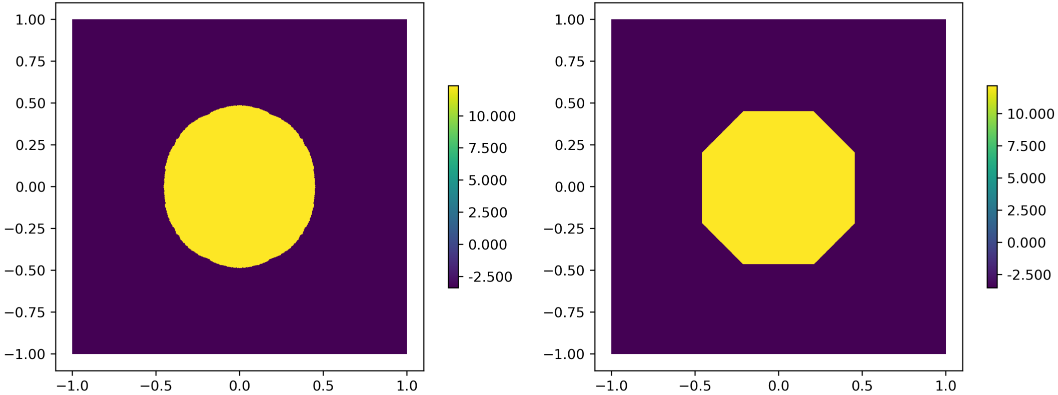
$$\begin{cases} -\Delta y = u & in\ \Omega \\ y = 0 & on\ \partial\Omega. \end{cases}$$

In the literature: $d = 2$ (Hafemeyer, Mannel, 2022), $d = 3$ (C.,Igleasias, Walter, 2023+)

Physical interpration $\approx$ to warm up a precise square in a room, with a sudden change of temperature outside the square.

Numerical example: the castle

Setup: $\alpha = 0.001$ with random (left) or regular (right) triangulation (500k) in $d = 2$.



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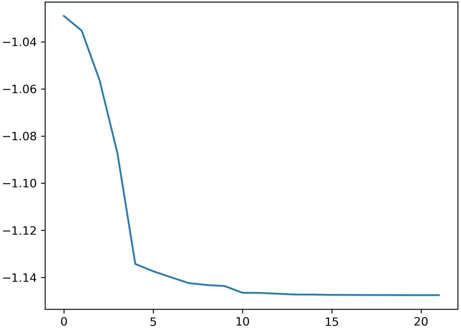
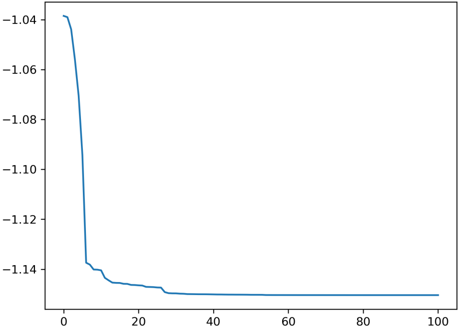
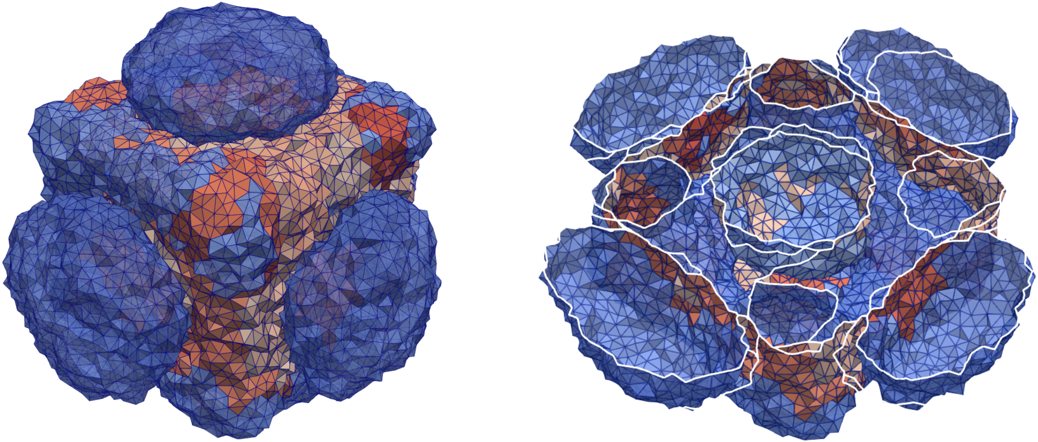




horizontal axis: iterations GCG. *vertical axis* energy values in $\log_{10}$ scale

Numerical example: the hypercastle

Setup: $\alpha = 0.001$ with random triangulation (800k) in $d = 3$.



Finale: discrete to continuous convergence

Let $\mathcal{T}_h$ be a sequence of triangulations in Ω, with $h \to 0$. Given $\varphi : S^{d-1} \to \mathbb{R}$, define

$$TV_\varphi(u) := \int_\Omega \varphi\left(\frac{Du}{|Du|}\right) d|Du|$$

in the sense of Radon-Nykodym (polar decomposition of Du w.r.t. $|Du|$).

Theorem (C., Iglesias, Walter)

If $P_{\mathcal{T}_h}$ Γ-converges (w.r.t. L^1 convergence of sets) to the anisotropic perimeter P_φ sending

$$A \mapsto \int_{\partial^* A} \varphi(\nu_A)\, d\mathcal{H}^{d-1}$$

then

$$F \circ K_h + \alpha TV_{\mathcal{T}_h} \xrightarrow{\Gamma - L^q} F \circ K + \alpha TV_\varphi$$

Look at (Chambolle, Giacomini, Lussardi 2009) for $TV_{\mathcal{T}_h} \xrightarrow{\Gamma - L^q} TV_\varphi$

Future work:

- Discrete to continuous convergence:

  $\rightarrow$ existence of φ for sequence of periodic triangulations

  $\rightarrow$ To describe φ for periodic triangulations ($\approx$ (Chambolle, Kreutz, 2022))

  $\rightarrow$ is it possible to have φ isotropic?

- Discrete minimizing movements for energies with surfaces and bulks:

$$\mathcal{E}(E, u) = \int_{\partial^* E} \varphi(x, \nu_E)\, d\mathcal{H}^{d-1} + \int_E W(\nabla u)$$

Thank you